Self-presentation.

1 First and last name

Adam Chudecki

2 Scientific degrees

1. **Master of Science** – Faculty of Technical Physics, Information Technology and Applied Mathematics, Lodz University of Technology, 2002, thesis: *Phase space of quantum systems*, advisor: dr hab. Jaromir Tosiek
2. **Doctor of Philosophy** – Faculty of Technical Physics, Information Technology and Applied Mathematics, Lodz University of Technology, 2009, thesis: *Plebański - Robinson - Finley hyperheavenly equations: analysis and applications in theory of relativity*, advisor: prof. dr hab. Maciej Przanowski

3 Employment in academic institutions

1. 2002–2004, assistant, Faculty of Technical Physics, Information Technology and Applied Mathematics, Lodz University of Technology
2. since 2004, assistant, Center of Mathematics and Physics, Lodz University of Technology

4 Scientific Achievement

4.1 Title of the scientific achievement — a monographic series of publications

Congruences of null strings and their relation with the symmetries in weak and strong hyperheavenly spaces.

4.2 The monographic series of publications

[H1] A. Chudecki, 2010, *Conformal Killing vectors in nonexpanding $\mathcal{HH}$-spaces with Λ*, Classical and Quantum Gravity **27**, 205004

[H2] A. Chudecki, 2012, *Classification of the Killing vectors in nonexpanding $\mathcal{HH}$-spaces with Λ*, Classical and Quantum Gravity **29**, 135010

[H3] A. Chudecki, 2013, *Homothetic Killing vectors in expanding $\mathcal{HH}$-spaces with Λ*, International Journal of Geometric Methods in Modern Physics, Vol. 10, No. 1, 1250077

[H4] A. Chudecki, 2014, *Null Killing vectors and geometry of null strings in Einstein Spaces*, General Relativity and Gravitation **46**, 1714

[H5] A. Chudecki, 2017, *Classification of the traceless Ricci tensor in 4-dimensional pseudo-Riemannian spaces of neutral signature*, Acta Physica Polonica B, Vol. 48, No. 1, 53-74

[H6] A. Chudecki, 2017, *On some examples of para-Hermite and para-Kähler Einstein spaces with $\Lambda \neq 0$*, Journal of Geometry and Physics 112, 175–196

[H7] A. Chudecki, 2017, *On geometry of congruences of null strings in 4-dimensional complex and real pseudo-Riemannian spaces*, Journal of Mathematical Physics **58**, 112502

[H8] A. Chudecki, 2018, *Classification of complex and real vacuum spaces of the type $[N] \otimes [N]$*, Journal of Mathematical Physics **59**, 062503

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4.3 Descriptions of the scientific goal of the monographic series of publications and the results achieved and possible applications of the results

4.3.1 Introduction

Complex methods in General Theory of Relativity

General Theory of Relativity (GTR) was introduced by Albert Einstein in 1915. Since that time new methods of finding the solutions of the gravitational field equations have been sought after. Many of these methods are based on complex analysis. The spinorial formalism [36, 40], the null tetrad formalism [6] and the twistor formalism [37] are the most transparent examples. In [60] new solutions of the Maxwell equations have been found by complex transformations of the coordinates. The same method allowed to find new spacetimes from already known metrics [7, 30]. E. Newman considered complex 4-dimensional spaces which metrics satisfy the vacuum Einstein field equations and for which self-dual (SD) or anti-self-dual (ASD) part of the Weyl tensor vanishes [31, 32]. Such spaces have been called *heavenly spaces* ($\mathcal{H}$*-spaces*). J.F. Plebański proved [41] that in heavenly spaces Einstein equations reduce to a single equation for one holomorphic function of four variables which completely determines the metric. This nonlinear partial differential equation of the second order was called *heavenly equation.*

A special attention was paid on the technique of the *real slices*[1] of complex spaces (wide class of the metrics was obtained as a Lorentzian slice of the double Kerr - Schild metric). Lorentzian slices of the complex metrics seemed to be a perfect tool for finding new solutions of Einstein equations. Many papers was devoted to this technique [43, 45, 54, 56, 62]. Especially interesting is the paper [56] in which K. Rózga analyzed the properties of the real slices. Obviously, the real slice technique allows to find not only Lorentzian spaces. *Riemannian spaces* equipped with the metric of the signature $(++++)$ and *neutral spaces* (also called *ultrahyperbolic*) equipped with the metric of the signature $(++--)$ can be also find by the real slice technique.

A basic corollary formulated in [56] was the necessary condition for Lorentzian slices to exist: SD and ASD parts of the Weyl tensor must be of the same Petrov - Penrose type[2]. Hence, heavenly spaces are useless for investigations of the Lorentzian slice technique. Heavenly spaces are of the types $[-]\otimes[\text{any}]$ or $[\text{any}]\otimes[-]$ and they admit only conformally flat Lorentzian slices.

Hyperheavenly spaces

Transparent progress was made in 1976. J.F. Plebański and I. Robinson introduced generalization of the heavenly spaces, so called *hyperheavenly spaces* [46, 47].

Definition 4.1. *Hyperheavenly space*[3] *with cosmological constant* Λ *(*$\mathcal{HH}$*-space with* Λ*) is a 4-dimensional complex analytic manifold equipped with the holomorphic metric which satisfies the vacuum Einstein field equations with cosmological constant and such that SD (or ASD) part of the Weyl tensor in algebraically special.*

Hyperheavenly spaces are the spaces of the types $[\text{deg}]\otimes[\text{any}]$ or $[\text{any}]\otimes[\text{deg}]$. In what follows we assume that SD part of the Weyl tensor is algebraically special. We also use abbreviation *hyperheavenly space* instead of *hyperheavenly space with cosmological constant.* The transparent result was, that in the hyperheavenly spaces the vacuum Einstein equations with cosmological constant can be reduced to a single nonlinear partial differential equation of the second order for one holomorphic function which completely determines the metric.

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[1] *Real slice* of the complex space is a 4-dimensional real submanifold of this space.

[2] In complex, neutral and Riemannian spaces SD and ASD parts of the Weyl tensor are independent. Therefore, "mixed" types can occur. In complex and neutral spaces both parts of the Weyl tensor are arbitrary; there appear, e.g, spaces of the type $[\text{II}]\otimes[\text{N}]$. In Riemannian spaces SD and ASD parts of the Weyl tensor can be of the types [I], [D] or $[-]$ only.

[3] Analogously with the terminology used in [41], $\mathcal{HH}$ spaces with $\Lambda = 0$ are also called *strong hyperheavenly spaces.*

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There are two types of the hyperheavenly spaces. The first type are *nonexpanding hyperheavenly spaces*. The metric of such spaces has the form

$$ds^2 = 2\,(-dp^{\dot{A}}dq_{\dot{A}} + Q^{\dot{A}\dot{B}}\,dq_{\dot{A}}dq_{\dot{B}}),\ \ \dot{A},\dot{B} = \dot{1},\dot{2} \tag{IV.1}$$

where $(q_{\dot{A}}, p_{\dot{B}})$ are local coordinates and $Q^{\dot{A}\dot{B}}$ has the form

$$Q^{\dot{A}\dot{B}} = -\Theta_{p_{\dot{A}}p_{\dot{B}}} + \frac{2}{3}F^{(\dot{A}}p^{\dot{B})} + \frac{1}{3}\Lambda\, p^{\dot{A}}p^{\dot{B}} \tag{IV.2}$$

$\Theta = \Theta(q_{\dot{A}}, p_{\dot{B}})$ is a holomorphic function called *the key function* and it satisfies *nonexpanding hyperheavenly equation with* Λ

$$\begin{aligned}&\frac{1}{2}\Theta_{p_{\dot{A}}p_{\dot{B}}}\Theta_{p^{\dot{A}}p^{\dot{B}}} + \Theta_{p_{\dot{A}}q^{\dot{A}}} + F^{\dot{A}}\Big(\Theta_{p^{\dot{A}}} - \frac{2}{3}p^{\dot{B}}\,\Theta_{p^{\dot{A}}p^{\dot{B}}}\Big) + \frac{1}{18}\,(F^{\dot{A}}p_{\dot{A}})^2 \\ &+\frac{1}{6}\frac{\partial F_{\dot{A}}}{\partial q^{\dot{B}}}\,p^{\dot{A}}p^{\dot{B}} + \Lambda\Big(p^{\dot{A}}\Theta_{p^{\dot{A}}} - \Theta - \frac{1}{3}p^{\dot{A}}p^{\dot{B}}\Theta_{p^{\dot{A}}p^{\dot{B}}}\Big) = N_{\dot{A}}\,p^{\dot{A}} + \gamma\end{aligned} \tag{IV.3}$$

$F^{\dot{A}}$, $N^{\dot{A}}$ and γ are arbitrary functions of $q^{\dot{C}}$. These functions are related to the coefficients of the SD conformal curvature [5, 16].

The second type of the hyperheavenly spaces are *expanding hyperheavenly spaces*. The metric of such spaces takes the form

$$\begin{aligned}ds^2 = (\phi\tau)^{-2}\Big\{&2\tau(d\eta dw - d\phi dt) + 2\,\Big(-\tau^2\phi\,W_{\eta\eta} + \mu\phi^3 + \frac{\Lambda}{6}\Big)dt^2 \\ &+4\tau^2\,(W_\eta - \phi\,W_{\eta\phi})\,dwdt + 2\tau^2\,(2\,W_\phi - \phi\,W_{\phi\phi})\,dw^2\Big\}\end{aligned} \tag{IV.4}$$

Coordinates (ϕ, η, w, t) are called *Plebański - Robinson - Finley coordinates (PRF coordinates)*. Function $W = W(\phi, \eta, w, t)$ is also called *the key function*. It satisfies *expanding hyperheavenly equation with* Λ

$$\begin{aligned}&\tau^2\Big(W_{\eta\eta}W_{\phi\phi} - W_{\eta\phi}W_{\eta\phi} + 2\phi^{-1}W_\eta W_{\eta\phi} - 2\phi^{-1}W_\phi W_{\eta\eta}\Big) + \tau\phi^{-1}\Big(W_{w\eta} - W_{t\phi}\Big) \\ &-\mu\Big(\phi^2W_{\phi\phi} - 3\phi W_\phi + 3W\Big) + \frac{\eta}{2\tau}(\mu_t\eta - \mu_w\phi) - \frac{\Lambda}{6}\phi^{-1}W_{\phi\phi} = \frac{1}{2}\varkappa\phi - \frac{1}{2}\nu\eta + \gamma\end{aligned} \tag{IV.5}$$

Functions μ, $\varkappa$, ν and γ are arbitrary functions of (w, t). Analogously like in the nonexpanding case they are related to the coefficients of the SD conformal curvature. τ is an arbitrary nonzero constant.

All analytic and algebraically special solutions of the vacuum Einstein equations are contained in the Lorentzian slices of the hyperheavenly spaces. The reduction of the vacuum Einstein equations to one hyperheavenly equation gave hope for progress of the very ambitious scientific programme called *Penrose - Plebański programme*. The main goal of this programme is finding the general techniques for obtaining Lorentzian slices of the complex spaces. Unfortunately, such techniques are still unknown. Even the examples of the Lorentzian slices of the complex metrics which give the famous metrics known in GTR are rare [4, 21]. A few examples have been found in [H3], [H4] and [H8].

Particularly, J.F. Plebański, J.D. Finley, M. Przanowski and others wanted to use the hyperheavenly spaces theory to find new vacuum type [N] solutions with twist[4]. Type [N] is the most algebraically degenerated of all nonconformally flat Lorentzian spaces. It is important in the theory of the gravitational waves. According to *Sachs peeling theorem* far away from a bounded source of gravitational radiation, the gravitational field is approximately of the Petrov-Penrose type [N]. All type [N] vacuum and nontwisting solutions are explicitly known

[4]Twist is one of the parameters which describe the optical properties of the congruences of null geodesics.

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(the pp-waves class, the Kundt class) or they have been reduced to a single PDE of the second order (the Robinson - Trautman class). Only one vacuum and twisting type [N] solution is explicitly known. It is *the Hauser solution* [23, 24]. Unfortunately, all explicitly known type [N] vacuum solutions have the singularities. Therefore, they cannot describe the gravitational waves. Despite of many approaches to the vacuum type [N] problem (e.g., [13, 14, 19], [P1] and [P9]), the Hauser solution is still the only known twisting solution.

The hyperheavenly equations (IV.3) and (IV.5) are strongly nonlinear. It is desired to investigate the ways of simplification of these equations. Obviously, the hyperheavenly spaces can be equipped with the symmetries defined by the Killing or homothetic vectors[5]. There are only two papers [42, 57] devoted to the symmetries in the hyperheavenly spaces. Symmetries in the heavenly spaces were considered in [15, 17].

Congruences of null strings

The hyperheavenly spaces are equipped with the interesting geometrical structure: *a congruence of null strings* [44]. The congruence of null strings (also called *the foliation of null strings*) is a family of complex 2-dimensional surfaces which are totally null and totally geodesic.

Consider 2-dimensional distribution $\mathcal{D}$ defined in an open subset $U \in \mathcal{M}$ by the Pfaff system

$$m_A g^{A\dot{B}} = 0\ , \quad (g^{A\dot{B}}) := \sqrt{2}\begin{bmatrix} e^4 & e^2 \\ e^1 & -e^3 \end{bmatrix}\ , \ A, B = 1, 2 \tag{IV.6}$$

where m_A is a nowhere vanishing 1-index undotted spinor field and (e^1, e^2, e^3, e^4) is *the null tetrad*, i.e., the basis of 1-forms such that the metric takes the form $ds^2 = 2e^1e^2 + 2e^3e^4$. Therefore, the distribution $\mathcal{D}$ is spanned by the vectors $\{m_A a_{\dot{B}}, m_A b_{\dot{B}}\}$, $a_{\dot{B}} b^{\dot{B}} \neq 0$. If one defines the 2-form Σ according to the equation $\Sigma := (m_A g^{A\dot{1}}) \wedge (m_B g^{B\dot{2}})$, then it takes the form

$$\Sigma = m_A m_B S^{AB} \tag{IV.7}$$

where S^{AB} is the basis of the SD 2-forms

$$(S^{AB}) := \begin{bmatrix} 2e^4 \wedge e^2 & e^1 \wedge e^2 + e^3 \wedge e^4 \\ e^1 \wedge e^2 + e^3 \wedge e^4 & 2e^3 \wedge e^1 \end{bmatrix} \tag{IV.8}$$

Because 2-form Σ is SD we say that the distribution $\mathcal{D}$ is SD. The distribution $\mathcal{D}$ is completely integrable in the Frobenius sense if the spinor m_A satisfies the set of the equations

$$m^A m^B \nabla_{A\dot{M}} m_B = 0 \tag{IV.9}$$

Eqs. (IV.9) are called *SD null string equations.* One also says that spinor m_A *generates the congruence of SD null strings* if it satisfies Eqs. (IV.9). *Null strings* are the integral manifolds of the distribution $\mathcal{D}$. The family of such integral manifolds constitutes *the congruence of null strings.* Every surface of this family is totally null and totally geodesic. [Analogously the congruences of ASD null strings are defined].

The properties of the congruences of null strings were investigated in [48,50,55]. Particularly, from the Eqs. (IV.9) it follows that

$$\nabla_{A\dot{M}} m_B = Z_{A\dot{M}} m_B + \in_{AB} M_{\dot{M}} \tag{IV.10}$$

where $Z_{A\dot{M}}$ is *the Sommers vector* and $M_{\dot{M}}$ is the *expansion of the congruence of SD null strings.* The expansion is the most important characteristic of the congruences of null strings.

[5]We use the following terminology: a vector K_a which satisfies the set of the equations $\nabla_{(a}K_{b)} = \chi\, g_{ab}$ is called *the Killing vector,* if $\chi = 0$; *the homothetic vector* if $\chi =$ const; *the proper homothetic vector* if $\chi =$ const $\neq 0$; *the proper conformal vector* if $\chi \neq$ const.

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If $M_{\dot{M}} = 0$ the congruence is called *nonexpanding*. If $M_{\dot{M}} \neq 0$ one deals with the *expanding* congruence. If the congruence is nonexpanding then the distribution $\mathcal{D}$ is *parallely propagated*. It means that $\nabla_Y X \in \mathcal{D}$ for every vector $X \in \mathcal{D}$ and for every vector Y.

The existence of the congruences of SD (ASD) null strings was related to the algebraic degeneration of the SD (ASD) part of the Weyl tensor. The *complex Goldberg - Sachs theorem* says [44, 52].

Theorem 4.2 (Plebański, Hacyan, [44]). *In a complex Einstein space the following statements are equivalent*

- *space admits a congruence of SD null strings generated by the spinor m^A*
- *SD Weyl spinor is algebraically degenerated and spinor m^A is a multiple Penrose spinor*

■

Expanding (nonexpanding) hyperheavenly spaces are equipped with expanding (nonexpanding) congruences of null strings. According to the complex Goldberg - Sachs theorem the number of different congruences of null strings is equal to the number of the multiple Penrose spinors. Hyperheavenly spaces of the types [II,III,N] $\otimes$ [any] are equipped with only one congruence of SD null strings, while the type [D] $\otimes$ [any] is equipped with two congruences of null strings. However, the type [D] $\otimes$ [any] does not admit the existence of the two congruences of SD null strings such that one of them is expanding and the second one is nonexpanding [P4]. Consequently, the possible Petrov - Penrose types of the hyperheavenly spaces are:

- $[\mathrm{II}]^n \otimes$ [any], $[\mathrm{D}]^{nn} \otimes$ [any] - nonexpanding hyperheavenly spaces with $\Lambda \neq 0$
- $[\mathrm{III,N}]^n \otimes$ [any] - nonexpanding hyperheavenly spaces with $\Lambda = 0$
- $[\mathrm{II,III,N}]^e \otimes$ [any], $[\mathrm{D}]^{ee} \otimes$ [any] - expanding hyperheavenly spaces

Upper index e means, that the congruence of SD null strings is expanding, while upper index n means, that the congruence of SD null strings is nonexpanding. If there are two different congruences then we use two indices, nn or ee.

There are infinitely many different congruences of SD null strings in the heavenly spaces of the types $[-] \otimes$ [any]. However, if $\Lambda = 0$ there exist both expanding and nonexpanding congruences. If $\Lambda \neq 0$, there exist only expanding congruences. Usually in such cases the upper index is omitted, unless one of the congruences is somehow distinguished and it is desired to point out its properties (see, e.g., [H4]).

Neutral spaces

It should be mentioned that real and totally null surfaces have been known in mathematics in *Walker spaces* since the fifties [27, 28, 61]. The hyperheavenly spaces formalism allowed for a transparent progress in this field of science [P2]. It is quite a difficult task to find the Lorentzian slice of the complex space, but it is quite easy to find the real neutral slice of the complex space. It is enough to find the null tetrad which all members can be considered as real ones. In many cases it is sufficient to replace all the complex coordinates by real ones and all the holomorphic functions by real analytic ones. Therefore, the hyperheavenly spaces are very useful tool in finding real neutral Einstein spaces.

Last times real neutral spaces appeared in many issues of the theoretical physics. In [22] "sharp" versions of the Goldberg - Sachs theorem were considered. Real neutral spaces equipped with two different congruences of null strings appeared in the papers devoted to the two solids which roll on each other without slipping and twisting [33,34]. Particularly, neutral ASD spaces should be mentioned. Such spaces appear in the Osserman geometry [1, 2, 20]. Real neutral spaces equipped with the Killing vectors were also considered [26], [P5], [P7]. The necessary

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and sufficient conditions for the 4-dimensional ASD spaces to be locally conformally equivalent to the Einstein spaces were analyzed in [11]. Summing up, real 4-dimensional neutral spaces have attracted a great deal of interest.

Sketch of the self-presentation

The monographic series of publications [H1] - [H8] is devoted to the use of the symmetries and congruences of null strings in studies of complex and real spaces. The most important results of this series of publication I am going to present in further parts of the self-presentation.

The papers [P1] - [P3] which were published before I received the Ph.D. degree proved that the hyperheavenly formalism is very useful. The example of the space of the type $[\mathrm{N}] \otimes [\mathrm{N}]$ equipped with the twisting congruence of null geodesics was found in [P1]. This example did not posses the Lorentzian slice, but it gave a new hope in progress of the further studies on such spaces. We decided that the analysis of the space of the type $[\mathrm{N}] \otimes [\mathrm{N}]$ with the symmetry defined by the homothetic vector should be the next step of our investigations. In [P2] the hyperheavenly spaces formalism allowed to obtain the explicit metrics of the 4-dimensional Walker and two-sided Walker spaces. Explicit examples of the metrics of the Osserman spaces which are not the Walker spaces were the main result of [P3]. The cosmological constant played a fundamental role in obtaining the examples of such spaces.

The results of [P1] - [P3] suggested that the thorough analysis of the symmetries of the hyperheavenly spaces with nonzero cosmological constant is strongly desired. In 2009 I formulated the plan of my scientific work. The main goal of this plan was the generalization of the results of [17, 42, 57] for the case of nonzero cosmological constant and for the case of proper conformal vectors. This plan was fulfilled in 2010-2013. The results were published in [H1] - [H3]. From these investigations it followed that there is a relation between null homothetic vectors and congruences of null strings[6]. The analysis of all the complex and real Einstein spaces equipped with the null homothetic vector was published in [H4].

The results of [P5] and [P6] (which are not a part of the monographic series of publications) convinced me that the existence of the congruences of null strings plays a very important role in the complex and real spaces. In 2015 I defined the second part of my plan of the scientific work. The main goal of this plan was to analyze how the existence of the congruences of null strings affects on the properties of the space. Firstly, I used the congruence of null strings as a tool in finding the explicit examples of the para-Hermite and para-Kähler Einstein spaces. The results were published in [H6]. Then I considered the existence of the congruences of null strings in algebraically degenerated spaces which are not the Einstein spaces (such spaces are called *weak hyperheavenly spaces*). I noticed, that the existence of such geometrical structures has a transparent affect on the algebraical properties of the traceless Ricci tensor. The classification of the traceless Ricci tensor in 4-dimensional neutral spaces was presented in [H5]. In [H7] the properties of the traceless Ricci tensor of the spaces equipped with the congruences of null strings were published. In the last paper [H8] I dealt with the vacuum type $[\mathrm{N}] \otimes [\mathrm{N}]$ spaces. Also, some new examples of the Lorentzian slices of the complex metrics were found in [H8].

4.3.2 Symmetries in the hyperheavenly spaces

The papers [H1] - [H2] are devoted to the symmetries in nonexpanding hyperheavenly spaces. The main goal of these papers was to fill the gaps left by J.F. Plebański and J.D. Finley in [42], i.e., the generalization of the analysis to the case of proper conformal symmetries and nonzero cosmological constant.

Nonexpanding hyperheavenly spaces with $\Lambda \neq 0$ are of the types $[\mathrm{II}]^n \otimes [\mathrm{any}]$ or $[\mathrm{D}]^{nn} \otimes [\mathrm{any}]$. If cosmological constant vanishes then nonexpanding hyperheavenly spaces are of the types $[\mathrm{III,N}]^n \otimes [\mathrm{any}]$. The generalization of the analysis to the case with nonzero cosmological constant

[6] It appeared later, that such relation has been already known, compare [12].

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is especially important, because symmetries in the nonexpanding hyperheavenly spaces of the types $[\mathrm{II}]^n \otimes [\mathrm{any}]$ and $[\mathrm{D}]^{nn} \otimes [\mathrm{any}]$ were not considered by J.F. Plebański and co-workers.

The first step was to demonstrate that the Killing equations $\nabla_{(a}K_{b)} = \chi g_{ab}$ can be reduced to a single equation. I called this equation *the master equation* (analogously, like the authors of the paper [42] did). In [H1] I considered different algebraic types separately[7]. I divided the analysis into three subsections (4.1, 4.2 i 4.3) in which I dealt with the types $[-] \otimes [\mathrm{any}]$, $[\mathrm{III,N}]^n \otimes [\mathrm{any}]$ and $[\mathrm{II,D}]^n \otimes [\mathrm{any}]$. The most important results are listed below.

- The results obtained for the heavenly spaces of the types $[-] \otimes [\mathrm{any}]$ are especially interesting. Analysis was generalized to the case of the proper conformal symmetries. I found the same embarrassing detail mentioned by J.F. Plebański and J.D. Finley in the paper [42]: there appeared the first integral of the heavenly equation Υ in the master equation. It should me mentioned, that in the case of the homothetic symmetries the function Υ can be gauged away by a clever trick introduced in the paper [17]. However, for the proper conformal symmetries this trick does not work. I realized, that the proper conformal symmetries in the heavenly spaces require a separate paper[8].
- Analysis of the types $[\mathrm{II}]^n \otimes [\mathrm{any}]$ and $[\mathrm{D}]^{nn} \otimes [\mathrm{any}]$ is original (for such types cosmological constant $\Lambda \neq 0$). The reduction of the Killing equations to a single equation was presented in details (section 6).
- The example considered in section 5 is devoted to the less algebraically degenerated hyperheavenly space which admits proper conformal vector. It is space of the type $[\mathrm{N}]^n \otimes [\mathrm{N}]^n$. The existence of a proper conformal vector implies the existence of a null Killing vector (it is covariant derivative of the conformal factor $\nabla_a \chi$). I solved the master equations for the proper conformal vector and for the null Killing vector and the hyperheavenly equation. The final result is the metric (5.27)[9]. It should be mentioned that in this example the explicit form of the key function was found. The function which generates the metric is given by Eq. (5.26).

I realized that my approach to the reduction of the Killing equations is quite unnatural (the considerations were divided into different algebraic types which were considered separately). I returned to this problem in [H2]. I proved that any Killing vector or homothetic vector or proper conformal vector in the nonexpanding hyperheavenly spaces of the types $[\mathrm{II,III,N}]^n \otimes [\mathrm{any}]$ or $[\mathrm{D}]^{nn} \otimes [\mathrm{any}]$ has the form

$$K = \delta^{\dot{B}} \frac{\partial}{\partial q^{\dot{B}}} + \left(2\chi p^{\dot{B}} + \frac{\partial \delta^{\dot{M}}}{\partial q_{\dot{B}}} p_{\dot{M}} + \epsilon^{\dot{B}} \right) \frac{\partial}{\partial p^{\dot{B}}} \tag{IV.11}$$

and the Killing equations can be reduced to the single master equation

$$\begin{aligned} \pounds_K \Theta = 2\Theta \left(3\chi - \frac{\partial \delta^{\dot{N}}}{\partial q^{\dot{N}}} \right) + \frac{1}{6} \frac{\partial^2 \delta_{\dot{A}}}{\partial q^{\dot{B}} \partial q^{\dot{C}}} p^{\dot{A}} p^{\dot{B}} p^{\dot{C}} \\ + \left(\frac{1}{3} F_{\dot{A}} \epsilon_{\dot{B}} + \frac{1}{2} \frac{\partial \epsilon_{\dot{A}}}{\partial q^{\dot{B}}} \right) p^{\dot{A}} p^{\dot{B}} + \zeta_{\dot{A}} p^{\dot{A}} + \xi \end{aligned} \tag{IV.12}$$

where $\pounds_K$ is the Lie derivative along a vector field K and functions $\delta^{\dot{A}}$, $\epsilon^{\dot{A}}$, $\zeta^{\dot{A}}$ and ξ are arbitrary functions of the coordinates $q^{\dot{M}}$. The integrability conditions of the Killing equations are given by the formulas (2.19), (2.20) and (2.21) in [H2].

The detailed classification of Killing vectors in the nonexpanding hyperheavenly spaces is the main result of [H2]. I analyzed the form of the Killing vectors and I solved the master equation in all the cases. I found the forms of the spinors[10] l_{AB} and $l_{\dot{A}\dot{B}}$ and I presented the form of

[7] There are several misprints in [H1]; these misprint have been corrected in the erratum in [H3].
[8] Together with M. Dobrski we devoted to this subject the paper [P6] which was published in 2014.
[9] This metric is a special case of the complex pp-wave metric which was considered in [H2].
[10] These spinors are proportional to the SD and ASD parts of the 2-form $\nabla_{[a}K_{b]}$.

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the reduced nonexpanding hyperheavenly equation (these results are contained in section 3). I proved that the nonexpanding hyperheavenly spaces admit three types of the Killing vectors ($\partial_{q^{\dot{1}}}$, $q^{\dot{1}}\partial_{p^{\dot{1}}}$, $\partial_{p^{\dot{1}}}$) and two types of the homothetic vectors ($\partial_{q^{\dot{1}}} + 2\chi_0 p^{\dot{A}}\partial_{p^{\dot{A}}}$, $2\chi_0 p^{\dot{A}}\partial_{p^{\dot{A}}}$).

The classification of the proper homothetic symmetries in the heavenly spaces were considered in subsection 3.6. Standard approach to this issue distinguishes three types of such symmetries. These types depend on the properties of the spinor l_{AB}:

- $l^{AB}l_{AB} \neq 0$ (compare [3, 10])
- $l^{AB}l_{AB} = 0$ (compare [9])
- $l_{AB} = 0$ (compare [12]) - in this case the proper homothetic vector is null

My approach distinguishes four types of such symmetries. The case $l^{AB}l_{AB} = 0$ corresponds to the type which I called $\mathcal{H}$HKI. The case $l_{AB} = 0$ corresponds to the type $\mathcal{H}$HKIIIb. However, the case $l^{AB}l_{AB} \neq 0$ splits into two different subtypes. $\mathcal{H}$HKII and $\mathcal{H}$HKIIIa.

I found a few examples of the nonexpanding hyperheavenly spaces admitting the symmetries in [H2]. The most interesting ones are:

- The metric (4.4) which is the general metric which admits the null Killing vector $K = q^{\dot{1}}\partial_{p^{\dot{1}}}$. The metric (4.4) is of the types $[\mathrm{III}, \mathrm{N}, -]^n \otimes [\mathrm{N}, -]^e$.
- The metric (4.6) which is the general metric which admits the null Killing vector $K = \partial_{p^{\dot{1}}}$. The metric (4.6) is of the type $[\mathrm{N}, -]^n \otimes [\mathrm{N}, -]^n$ and in the next papers it was called *the complex pp-wave metric*. This metric admits real Lorentzian slice which is vacuum type [N] pp-wave metric. In the hyperheavenly spaces formalism the key function which determines pp-waves metric is given by the formula (4.10). It is the first example of the Lorentzian slice of the complex metric I have found.
- The metric (4.15) which is the example of the space of the type $[\mathrm{N}]^n \otimes [\mathrm{N}]^n$ with the nonnull Killing vector of the type $K = \partial_{q^{\dot{1}}}$.
- The metric (4.18) which is the example of the space with $\Lambda \neq 0$ of the quite rare type $[\mathrm{II}]^n \otimes [\mathrm{N}]^e$.

The paper [H3] is devoted to the symmetries in the expanding hyperheavenly spaces. The main goal of this paper was to generalize the results obtained by A. Sonnleitner and J.D. Finley in [57] for the case with nonzero cosmological constant. Note, that the expanding hyperheavenly spaces do not admit proper conformal vectors. I proved, that any Killing or homothetic vector in the spaces of the types $[\mathrm{II}, \mathrm{III}, \mathrm{N}]^e \otimes$ [any] or $[\mathrm{D}]^{ee} \otimes$ [any] takes the form

$$K = a\,\frac{\partial}{\partial w} + b\,\frac{\partial}{\partial t} + (b_t - 2\chi_0)\phi\,\frac{\partial}{\partial \phi} + \Big((2b_t - a_w - 2\chi_0)\eta + b_w\phi - \tau\epsilon\Big)\frac{\partial}{\partial \eta} \tag{IV.13}$$

and the Killing equations reduce to the master equation

$$\begin{aligned} \pounds_K W \; = \; & -(4\chi_0 + 2a_w - 3b_t)W + \frac{b_w}{2\tau^2}\eta\Big(\mu\phi^3 - \frac{\Lambda}{3}\Big) + \alpha\phi^3 \\ & + \frac{1}{2\tau}\Big(- b_{ww}\phi^2 - b_{tt}\eta^2 + (a_{ww} - 2b_{tw})\eta\phi\Big) + \frac{1}{2}(\epsilon_w\phi + \epsilon_t\eta) + \beta \end{aligned} \tag{IV.14}$$

where b, ϵ, α and β are arbitrary functions of the coordinates (w, t) and $a = a(w)$. The integrability conditions of the Killing equations are given by the formulas (3.26a) and (3.35a) - (3.35e).

The classification of the symmetries is presented in section 4. I analyzed the forms of the Killing and homothetic vectors and the reductions of the hyperheavenly equation for all

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symmetries and all algebraic types. The expanding hyperheavenly spaces admit three types of the Killing vectors (∂_w, ∂_t, ∂_η) and three types of the homothetic vectors ($\partial_w - 2\chi_0(\phi\partial_\phi + \eta\partial_\eta)$, $\partial_t - 2\chi_0(\phi\partial_\phi + \eta\partial_\eta)$, $-2\chi_0(\phi\partial_\phi + \eta\partial_\eta)$).

The most interesting examples obtained in section 5 are:

- Spaces of the type $[\mathrm{D}]^{ee} \otimes [\mathrm{any}]$ with the symmetries (subsection 5.1)
- The metric (5.23) which is the general metric for the types $[\mathrm{III}]^e \otimes \mathrm{III}]^e$ and $[\mathrm{N}, -]^e \otimes \mathrm{N}, -]^e$ with $\Lambda \neq 0$ admitting the null Killing vector ∂_η. The examples of pointwise Osserman and globally Osserman spaces which are not Walker spaces can be easily obtained from the metric (5.23).

In all examples of the metrics admitting the null homothetic vector which were presented in [H2] and [H3] the null homothetic vector is tangent to the null string. Natural question arises: does the existence of the null vector imply the existence of the congruence of null strings? I decided to analyze the hyperheavenly and heavenly spaces with the null homothetic vector in details. I devoted to this issue the paper [H4].

It appeared that if one assumes the existence of the null homothetic vector (which can be always written in the form $K_{A\dot B} = m_A m_{\dot B}$) then from the integrability conditions of the Killing equations it follows that spinors m_A and $m_{\dot A}$ generate the congruences of SD and ASD null strings. Hence, any null homothetic vector is tangent to the SD and ASD null string. It appeared later that this result has been already known, at least in the ASD spaces (compare [12]). The general analysis presented in [H4] leads to the formulas (2.31a)-(2.31f). The formula for the covariant derivative of the null homothetic vector is especially interesting

$$\nabla_A^{\ \dot B} K_C^{\ \dot D} = m_C M_A \in^{\dot B \dot D} + m^{\dot D} M^{\dot B} \in_{AC} \tag{IV.15}$$

spinors M_A and $M_{\dot B}$ are expansions of the congruences of ASD and SD null strings, respectively. The covariant derivative of such a vector is determined by four spinor fields m_A, $m_{\dot B}$, M_A and $M_{\dot B}$.

Then I found that null and proper homothetic vector is admitted only by the spaces of the types $[\mathrm{N}, -]^e \otimes [\mathrm{III}, -]^n$. Using the hyperheavenly spaces formalism I found the general metric of the space of the type $[\mathrm{III}]^n \otimes [\mathrm{N}]^e$ with the null and proper homothetic vector. It is the metric (4.6). There appear a function of three variables in this metric. This function satisfies the equation (4.5). Unfortunately, I was unable to solve the equation (4.5). Note, that the space of the type $[\mathrm{III}]^n \otimes [\mathrm{N}]^e$ equipped with the null and proper homothetic symmetry appeared earlier in [H2] and [H3] but it was not analyzed there.

The only non conformally flat heavenly spaces which admit null and proper homothetic vector are spaces of the types $[\mathrm{N}]^e \otimes [-]^n$ (the metric (4.9)) and $[\mathrm{III}]^n \otimes [-]^e$ (the metric (4.14)). These metrics are original results. To the best my knowledge the metrics of the heavenly spaces with null and proper homothetic symmetry have not been found earlier explicitly. Moreover, the metric (4.14) belongs to the class of two-sided Walker spaces.

There are a lot of metrics which admit null Killing vector. These are the metrics of the spaces of the types

$$\begin{array}{rl}
(i) & [\mathrm{N}, -]^n \otimes [\mathrm{N}, -]^n, \ \Lambda = 0 \\
(ii) & [\mathrm{III{,}N}, -]^n \otimes [\mathrm{N}, -]^e, \ \Lambda = 0 \\
(iii) & [\mathrm{III}]^e \otimes [\mathrm{III}]^e, \ [\mathrm{N}, -]^e \otimes [\mathrm{N}, -]^e, \ \Lambda \neq 0 \\
(iv) & [\mathrm{II}]^e \otimes [\mathrm{II}]^e, \ [\mathrm{D}]^{ee} \otimes [\mathrm{D}]^{ee}, \ \Lambda \text{ is arbitrary}
\end{array}$$

Some of the metrics which admit null Killing vector appeared in previous publications, e.g., (i) and (ii) appeared in [H2] and (iii) in [H3]. Note, that (i) admits the Lorentzian slice which is the pp-wave metric. Detailed analysis proved that the spaces (ii) and (iii) do not admit

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Lorentzian slices. The reason why *(ii)* does not admit such a slice lies in the different properties of the congruences of SD and ASD null strings (SD congruence is nonexpanding and ASD one is expanding). However, the case *(iii)* is much more subtle. Two necessary conditions for the Lorentzian slices to exist are satisfied by *(iii)* (both SD and ASD parts of the Weyl tensor are of the same Petrov - Penrose type and both congruences of null strings are expanding, i.e., they have the same properties). We know, however, that Lorentzian slices of such spaces do not exist. If they existed, they would have to be Einstein spaces of types [III] or [N] with $\Lambda \neq 0$, equipped with the null Killing vector. Such metrics do not exist in GTR.

The metric of the case *(iv)* (which corresponds to the type $[\mathrm{II}]^e \otimes [\mathrm{II}]^e$ with $\Lambda \neq 0$) was reduced to the form (5.5). It depends on one function $W(\phi, w, t)$ which satisfies the equation (5.6). The general solution of the equation (5.6) is unknown. Moreover, I was not able to find the transformation of the variables which brings the metric (5.5) to the form more plausible to obtain the Lorentzian slice (such a slice exists, compare [58]). I proved that the only possible algebraic reduction of the metric (5.5) gives the type $[\mathrm{D}]^{ee} \otimes [\mathrm{D}]^{ee}$ which corresponds to the trivial solution of the equation (5.6), namely $W = 0$. It is given by the metric (5.18).

However, I succeeded in finding the Lorentzian slice of the metric of the type $[\mathrm{II}]^e \otimes [\mathrm{II}]^e$ with $\Lambda = 0$. I found the transformation of the variables which brought the metric to the form

$$ds^2 = -2x\, du(dv + M du) + x^{-\frac{1}{2}}(dx^2 \pm dy^2) \tag{IV.16}$$

where function $M = M(x, y, u)$ satisfies Euler - Poisson - Darboux equation

$$xM_{xx} \pm xM_{yy} + M_x = 0 \tag{IV.17}$$

If the coordinates in (IV.16) and (IV.17) are considered as real ones and the function M as real analytic one, then upper signs correspond to the Lorentzian slice (i.e., the type [II] with $\Lambda = 0$ and with the null Killing vector). Lower signs correspond to the real neutral slice[11]. It is the second example of Lorentzian slice of a complex metric I was able to find.

4.3.3 Para-Hermite and para-Kähler Einstein spaces

In [P4] together with M. Przanowski and S. Formański, we analyzed *complex para-Hermite Einstein spaces*.

Definition 4.3. *Complex 4-dimensional para-Hermite space is a complex space equipped with nondegenerate holomorphic metric such that for every point p there exist an open neighborhood* $U \subset \mathcal{M}$ *and complex coordinates* $\{z^A, z^{\dot B}\}$ *such that*

$$ds^2 = 2 f_{A\dot B} dz^A dz^{\dot B}, \ \det(f_{A\dot B}) \neq 0 \tag{IV.18}$$

where $f_{A\dot B}$ *are holomorphic functions*[12].

Para-Hermite spaces are equipped with two different congruences of SD (or ASD) null strings. In general, both these congruences are expanding. If both congruences are nonexpanding then there exist a function f such that $f_{A\dot B} = \partial^2 f / \partial z^A \partial z^{\dot B}$. Such spaces are called *para-Kähler*. From the complex Goldberg - Sachs theorem it follows that para-Hermite Einstein spaces are the spaces of the type $[\mathrm{D}]^{ee} \otimes [\text{any}]$ and para-Kähler Einstein spaces are spaces of the type $[\mathrm{D}]^{nn} \otimes [\text{any}]$. Einstein field equations in the spaces of the type $[\mathrm{D}]^{ee} \otimes [\text{any}]$ can be reduced

[11] Transformation (5.13) which was found in [H4] leads only to the Lorentzian slice. However, simple modification of the transformation (5.13) leads to the real neutral slice. Namely, it is enough to change the coordinate t according to $\frac{t}{\tau} = \left(-\frac{1}{\mu_0}\right)^{\frac{1}{3}}(x + y)$

[12] Note, that if the coordinates z_A and $z_{\dot B}$ are real, the corresponding real space is neutral; if $z_{\dot A} = \overline{z_A}$ then the corresponding real space has the metric of the signature $(++++)$.

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to a single equation and it is one of the most important results of [P4]. It is the equation (5.13) in [P4].

In [33, 34] neutral Einstein spaces with $\Lambda \neq 0$ play an important role. I decided to use the results of [P4] together with the hyperheavenly spaces formalism to find the explicit metrics of the complex and real neutral para-Hermite and para-Kähler spaces. The paper [H6] is devoted to this issue. I have chosen the null tetrad in such a manner that the first congruence of SD null strings was spanned by the vectors (∂_1, ∂_3) and the second congruence is spanned by the vectors (∂_2, ∂_4). However, it was clear for me that the existence of two different congruences of SD null strings is the condition not strong enough to obtain general solutions. I assumed then the existence of the additional congruence of ASD null strings (what is equivalent to the algebraic degeneration of the ASD Weyl spinor). There are two different ways of further investigations:

1. To use the hyperheavenly equation or the equation (5.13) in [P4]. However, it is impossible then to choose the null tetrad in such a manner that the congruence of ASD null strings is spanned by the vectors (∂_1, ∂_4), at least in general.

2. To choose the null tetrad in such a manner that the congruence of ASD null strings is spanned by the vectors (∂_1, ∂_4). In this case it is necessary to solve vacuum Einstein equations with Λ from the beginning, because both the hyperheavenly equation and the equation (5.13) in [P4] are not valid anymore, at least in general.

I decided to simplify the problem and I followed these two ways simultaneously, i.e., I tried to solve the hyperheavenly equation or the equation (5.13) in [P4], but at the same time I assumed that the congruence of ASD null strings is spanned by the vectors (∂_1, ∂_4). This simplification allowed to obtain many examples of the explicit metrics of the para-Kähler Einstein spaces (such spaces exist only if $\Lambda \neq 0$) and para-Hermite Einstein spaces (for such spaces Λ is arbitrary but I focused on the case with $\Lambda \neq 0$). The solutions which were found are gathered in the Tables 1 and 2. Remarks:

- The type $[\mathrm{D}]^{nn} \otimes [\mathrm{II}]^{n}$ I was able to solve later in all generality. This result was published in [P8] which is not the part of the monographic series of publications.

- The metric (5.13) which describes the type $[\mathrm{D}]^{nn} \otimes [\mathrm{D}]^{nn}$ is the general solution of the *para-Kähler homogeneous Einstein space*, compare [35]. It is at the same time the general solution of the type $[\mathrm{D}]^{nn} \otimes [\mathrm{D}]^{nn}$ what I proved later and the proof of this fact is not published in [H6].

- The types $[\mathrm{D}]^{ee} \otimes [\mathrm{III}]^{n}$ and $[\mathrm{D}]^{ee} \otimes [\mathrm{N}]^{n}$ exist only if $\Lambda = 0$.

- All the metrics of the type $[\mathrm{D}]^{ee} \otimes [\mathrm{D}]^{nn}$ found in [H6] do not admit any Lorentzian slices. To the best my knowledge these metrics are the first examples of the complex metrics for which SD and ASD Weyl spinors are of the type [D] and which do not admit Lorentzian slices.

4.3.4 Congruences of null strings in the weak hyperheavenly spaces

The results of [H6] showed that congruences of null strings in the hyperheavenly spaces are important geometrical structures. Other results [50, 55] proved that such structures play a distinguished role not only in Einstein spaces. The case of the para-Kähler space which is not the Einstein space was analyzed in [50]. An intriguing (though not widely known) result was presented in [55]. In this paper the general metric of the space which admits three different congruences of SD the null strings was found. Such a space cannot be Einstein since Einstein spaces admit at most two different congruences of SD null strings. I was curious if the existence of the congruences of null strings affects on the traceless Ricci tensor?

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Type	Examples
$[\mathrm{D}]^{nn} \otimes [\mathrm{I}]$	no examples
$[\mathrm{D}]^{nn} \otimes [\mathrm{II}]^e$	no examples
$[\mathrm{D}]^{nn} \otimes [\mathrm{II}]^n$	(5.11)
$[\mathrm{D}]^{nn} \otimes [\mathrm{D}]^{ee}$	after changing the orientation (3.43) for $S_0 = 0$ and (3.47) for $S_0 = 0$
$[\mathrm{D}]^{nn} \otimes [\mathrm{D}]^{nn}$	(5.13)
$[\mathrm{D}]^{nn} \otimes [\mathrm{III}]^e$	(5.16)
$[\mathrm{D}]^{nn} \otimes [\mathrm{III}]^n$	does not exist
$[\mathrm{D}]^{nn} \otimes [\mathrm{N}]^e$	(5.16) z $f = f_0 z$
$[\mathrm{D}]^{nn} \otimes [\mathrm{N}]^n$	does not exist

Table 1: Para-Kähler Einstein metrics found in [H6].

It was clear that a classification of the traceless Ricci tensor in 4-dimensional neutral spaces should be the first step. Such classification in Lorentzian case was presented in distinguished paper [39] (for a little different approach see [25, 38]). The paper [51] was devoted to such classification in complex spaces. I devoted the paper [H5] to the classification of the traceless Ricci tensor in the neutral spaces.

I focused on the algebraic properties of the matrix $(C^a_{\ b})$ of the traceless Ricci tensor and I used the following criteria:

- the number and the types of the eigenvectors (space-like, time-like, null)[13]
- the number and the type of the eigenvalues of the characteristic polynomial (complex or real, single, double, triple or quadruple)
- the form of the minimal polynomial
- Petrov - Penrose type of the Plebański spinors[14]

I distinguished 9 main types and 33 subtypes of the traceless Ricci tensor in the neutral spaces. So great number of the subtypes convinced me that it is worth to skip the elegant convention of the symbols of the types used by J.F. Plebański and M. Przanowski w [39, 51]. I used a little more complicated symbol of the type. Its advantage is that all the properties of the matrix $(C^a_{\ b})$ can be immediately guessed. I proposed the symbol

$$^{[\mathrm{A}_j]\otimes[\mathrm{B}_k]}[n_1 E_1^{\alpha_1} - n_2 E_2^{\alpha_2} - \ldots]^v_{(q_1 q_2 \ldots)}$$

where

- $[\mathrm{A}_j]$ and $[\mathrm{B}_k]$ are Petrov - Penrose types of the undotted and dotted Plebański spinors, respectively (Plebański spinors are symmetric in all indices. They can be classified like SD and ASD Weyl spinors. Note, that there are 10 such types in the neutral spaces, compare, e.g., [22])

[13] I defined space-like and time-like vector in the neutral spaces analogously as they are defined in the Lorentzian spaces: if $V^a V_a > 0$ the vector is space-like, if $V^a V_a < 0$ the vector is time-like.

[14] *Plebański spinors* are defined as follows: $V_{ABCD} := 4\,C_{(AB}{}^{\dot{M}\dot{N}}C_{AC)\dot{M}\dot{N}}$, $V_{\dot{A}\dot{B}\dot{C}\dot{D}} := 4\,C_{MN(\dot{A}\dot{B}}C^{MN}{}_{\dot{C}\dot{D})}$ where $C_{AB\dot{C}\dot{D}}$ is the spinorial image of the traceless Ricci tensor.

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Type	Examples
$[\mathrm{D}]^{ee} \otimes [\mathrm{I}]$	no examples
$[\mathrm{D}]^{ee} \otimes [\mathrm{II}]^{e}$	(3.31) for $f \neq 0$, (3.35) for $f_y \neq 0$
$[\mathrm{D}]^{ee} \otimes [\mathrm{II}]^{n}$	(3.31) for $f = 0$
$[\mathrm{D}]^{ee} \otimes [\mathrm{D}]^{ee}$	(3.43) for $S_0 \neq 0$ and (3.47) for $S_0 \neq 0$
$[\mathrm{D}]^{ee} \otimes [\mathrm{D}]^{nn}$	(3.43) for $S_0 = 0$ and (3.47) for $S_0 = 0$
$[\mathrm{D}]^{ee} \otimes [\mathrm{III}]^{e}$	(3.57)
$[\mathrm{D}]^{ee} \otimes [\mathrm{III}]^{n}$	(3.61)
$[\mathrm{D}]^{ee} \otimes [\mathrm{N}]^{e}$	no examples
$[\mathrm{D}]^{ee} \otimes [\mathrm{N}]^{n}$	no examples

Table 2: Para-Hermite Einstein metrics found in [H6].

- v is the number of the eigenvectors
- $(q_1 q_2 ...)$ describes the form of the minimal polynomial
- $E_i^{\alpha_i}$ are different eigenvalues, $E_i = \{Z, R\}$ (Z - complex, R - real), the upper index $\alpha_i = \{n, s, t, ns, nt, nst\}$ denotes the type of the corresponding eigenvector (n - null, s - space-like, t - time-like, ns - null or space-like, nt - null or time-like, nst - arbitrary)
- n_i are multiplicities of the eigenvalues

In [H5] I analyzed the criteria which distinguish the corresponding types, then I presented graphically the degeneration diagrams for parent types, finally I found the canonical forms of the $(C^a{}_b)$ for each type. After the classification was prepared, I returned to the question how to relate the properties of the congruences of null strings to the properties of the traceless Ricci tensor.

In [H7] I considered the spaces equipped with one, two, three and four different congruences of SD null strings. Two corollaries can be easily formulated from the integrability conditions of the null string equations:

1. if a spinor m_A generates a congruence of SD null strings, then it is a Penrose spinor (Theorem 3.3 in [H7])
2. if a spinor m_A generates a nonexpanding congruence of SD null strings, then it is a multiple Penrose spinor (Theorem 3.4 in [H7])

The spaces equipped with at least one nonexpanding congruence of SD null strings belong to the important class of *weak hyperheavenly spaces* (see, e.g., [P2]).

Definition 4.4. *Weak hyperheavenly space is a four-dimensional holomorphic space endowed with a holomorphic metric, which satisfies the following conditions:*

(i) *the SD Weyl spinor is algebraically special and spinor m_A is a multiple Penrose spinor*

(ii) *space admits a congruence of SD null strings generated by the spinor m_A*

Of course, in Einstein spaces $(i) \iff (ii)$ (it follows from the complex Goldberg - Sachs theorem). The most interesting results of [H7] are:

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- The existence of the congruences of SD null strings of different properties was related to the possible algebraic types of SD Weyl spinor (Tables I, IV, V and VII). Some of these results have been already known and some of them (section V) are original.
- It was proved that if the space admits two congruences of SD null strings then their expansions, Sommers vectors and the covariant derivatives of these objects determine the traceless Ricci tensor (formulas (4.7a)-(4.7c)).
- It was determined what types of the traceless Ricci tensor are admitted by the space equipped with one nonexpanding congruence of SD null strings (Table II). It is worth to point out the subtle difference between the types ${}^{(2)}[4N]_2^a$ and ${}^{(2)}[4N]_2^b$ described earlier in [50]. In the case of the type ${}^{(2)}[4N]_2^b$ both null eigenvectors of the traceless Ricci tensor are tangent to the null string. In the case of the type ${}^{(2)}[4N]_2^a$ only one eigenvector is tangent to the null string.
- It was determined what types of the traceless Ricci tensor are admitted by the space equipped with two nonexpanding congruences of SD null strings (Table VI).
- The metric obtained by I. Robinson and K. Rózga in [55] is the general metric of the space equipped with three different expanding congruences of SD null strings. This metric was specified for the case of two congruences being expanding and one congruence being nonexpanding (Theorem 5.3).

There is one more interesting result in [H7] which is related to the congruences of SD null strings but does not concern the weak hyperheavenly spaces anymore. In subsection V.C the space equipped with four different congruences of SD null strings was considered. It is possible only for the complex spaces of the type $[\mathrm{I}] \otimes [\mathrm{any}]$ and real neutral spaces of the type $[\mathrm{I}_r] \otimes [\mathrm{any}]$. Moreover, all congruences are necessarily expanding. The issue was reduced to the system of three equations for four functions (Theorem 5.4). I was not able, however, to solve this system[15].

4.3.5 Spaces of the type $[\mathrm{N}] \otimes [\mathrm{N}]$

Vacuum and twisting, type [N] Einstein equations are one of the unsolved problems of GR. We dealt with this problem together with M. Przanowski in [P1] and [P9]. We used the formalism of the hyperheavenly spaces of the type $[\mathrm{N}]^e \otimes [\mathrm{N}]^e$. We were not able, however, to obtain the explicit solutions. The first problem is that field equations for the space of the type $[\mathrm{N}]^e \otimes [\mathrm{N}]^e$ with twist are very complicated. The second problem is that we still do not known the general techniques of obtaining the Lorentzian slices of the complex spaces. Even if we succeed in finding a solution of the field equation or in reducing the problem into differential equation of the second order, there is still the question about the Lorentzian slice. The facts that both SD and ASD Weyl spinors are of the same algebraic type and the congruences of SD and ASD null strings have the same properties do not imply that the Lorentzian slice exists. I have already presented an example of the spaces with such properties which do not admit any Lorentzian slice (see page 10).

Therefore, it is desired to investigate the complex spaces which SD and ASD Weyl spinors are of the type [N] in details and reconstruct from these spaces all known Lorentzian type [N] solutions. Such an approach allows, perhaps, to formulate some general conclusions of the Lorentzian slices. Moreover, in [P9] extremely interesting metric of the type $[\mathrm{N}]^e \otimes [\mathrm{N}]^n$ equipped with the twisting null geodesics congruence and two homothetic vectors was found. The natural generalization of considerations of the paper [P9] is to analyze the same type

[15] I think that the problem of the spaces equipped with four different congruences of SD null strings can be explicitly solved, but it involves a different approach. It is one of the most important problems I am going to return to.

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without any symmetries and to investigate the field equations for this type. This question is studied in [H8].

In the first step in [H8] we assume the existence of the congruences of SD and ASD null strings. Then their properties were related to the properties of their mutual intersections. The intersections of the congruences of SD and ASD null strings constitute the congruence of null geodesics. I defined *complex expansion* θ, *complex twist* ϱ and *complex shear* s analogously as it is done in the Lorentzian spaces (formulas (3.6)). Then I considered the congruence of null geodesics in the affine parametrization and I found the relations between θ, ϱ and the expansions of the congruences of null strings

$$\begin{aligned} \theta &\sim m_A M^A + m_{\dot{A}} M^{\dot{A}} \\ \varrho &\sim m_A M^A - m_{\dot{A}} M^{\dot{A}} \end{aligned} \tag{IV.19}$$

where m_A is a spinor which generates the congruence of SD null strings with the expansion given by $M_{\dot{A}}$ and $m_{\dot{A}}$ is a spinor which generates the congruence of ASD null strings with the expansion given by M_A.

Any space of the type $[\mathrm{N}]\otimes[\mathrm{N}]$ can be classified according to the properties of the congruences of null strings. The properties of the congruence of null geodesics can be used as a subcriterion. There are exactly 6 different types of the Einstein type $[\mathrm{N}]\otimes[\mathrm{N}]$ spaces with $\Lambda = 0$ (see Table 3). The symbol $[++]$ means that the congruence of null geodesics is expanding and twisting, $[+-]$ means that the congruence is expanding and nontwisting, $[-+]$ means that the congruence is nonexpanding but twisting and finally, the symbol $[--]$ means that the congruence is nonexpanding and nontwisting. In the case of the spaces which are not Einstein spaces there exists one more type $\{[\mathrm{N}]^e\otimes[\mathrm{N}]^e, [-+]\}$. The existence of such a type in Einstein spaces with $\Lambda = 0$ is not possible (it follows from the Raychaudhuri equation).

Type	In [H8] considered in:
$\{[\mathrm{N}]^n\otimes[\mathrm{N}]^n, [--]\}$	section 4
$\{[\mathrm{N}]^e\otimes[\mathrm{N}]^n, [--]\}$	section 6
$\{[\mathrm{N}]^e\otimes[\mathrm{N}]^n, [++]\}$	section 8
$\{[\mathrm{N}]^e\otimes[\mathrm{N}]^e, [--]\}$	section 5
$\{[\mathrm{N}]^e\otimes[\mathrm{N}]^e, [+-]\}$	section 7
$\{[\mathrm{N}]^e\otimes[\mathrm{N}]^e, [++]\}$	not considered

Table 3: Possible type $[\mathrm{N}]\otimes[\mathrm{N}]$ spaces.

I found the key function for all the types and I inserted the key function into the hyperheavenly equation. For all the cases the hyperheavenly equation was completely solved or reduced to the PDE of the second order. I have investigated also the symmetries generated by one or two homothetic vectors in all the types. The most interesting results are listed below.

- The Lorentzian slice of the type $\{[\mathrm{N}]^e\otimes[\mathrm{N}]^e, [--]\}$ was found. It appeared that this Lorentzian slice is vacuum type [N] Kundt class (subsection 5.1). It is the third example of the Lorentzian slice I have found.

- It was proved that the complex pp-wave is the type $\{[\mathrm{N}]^n\otimes[\mathrm{N}]^n, [--]\}$ and the complex Kundt class is the type $\{[\mathrm{N}]^e\otimes[\mathrm{N}]^e, [--]\}$. In Lorentzian case both these metrics are equipped with the nonexpanding and nontwisting congruence of null geodesics. However, pp-wave metric admits the null Killing vector which is not admitted by the Kundt class.

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There is a transparent difference between the complexification of these spaces: complex pp-waves are equipped with two congruences of null strings which are both nonexpanding while the complex Kundt class is equipped with two expanding congruences. In both cases these congruences intersect one another along nontwisting and nonexpanding congruence of null geodesics.

- The Lorentzian slice of the type $\{[\mathrm{N}]^e \otimes [\mathrm{N}]^e, [+-]\}$ was found. It appeared that this Lorentzian slice is vacuum Robinson - Trautman class (subsection 7.1). It is the fourth example of the Lorentzian slice I have found.
- New class of the metrics of the type $\{[\mathrm{N}]^e \otimes [\mathrm{N}]^n, [--]\}$ was found. Such metrics admit only neutral slices[16] (section 6).
- New class of the metrics of the type $\{[\mathrm{N}]^e \otimes [\mathrm{N}]^n, [++]\}$ which admits only neutral slices and which is equipped with the twisting congruence of null geodesics was found (section 8). Within this class vacuum Einstein equations can be reduced to a single equation. It is Eq. (8.4) in the case with no symmetries and Eq. (8.9) in the case with one symmetry. The case with two symmetries was considered in [P9].

4.3.6 Summary of the monographic series of publications

The monographic series of publications which I have described in sections 4.3.2-4.3.5 is based on 8 articles [H1] - [H8] published in 2010-2018. These papers are devoted to the symmetries and geometry of the congruences of null strings in strong and weak hyperheavenly spaces. The most important results of the series of publications are:

- Detailed analysis of the Killing vectors, homothetic vectors and proper conformal vectors in the hyperheavenly spaces ([H1], [H2], [H3], [H4]).
- Classification of the traceless Ricci tensor in 4-dimensional neutral spaces ([H5]).
- Examples of the metrics of para-Hermite and para-Kähler Einstein spaces. Many of these metrics are - to the best my knowledge - the most general solutions of such spaces of certain algebraic types ([H6]).
- The analysis of the relation between geometrical properties of the congruences of null strings and the properties of the traceless Ricci tensor ([H7]).
- Detailed analysis of the spaces which both SD and ASD parts of the Weyl tensor are of the type [N] ([H8]).
- Four examples of the Lorentzian slices of the complex spaces. Three of them are Einstein spaces of the type [N] ([H2], [H8]) and one of them is an Einstein space of the type [II] ([H4]).

5 Description of other scientific achievements

5.1 Other publications

[P1] Chudecki A. i Przanowski M., 2008, *A simple example of type-[N] ⊗ [N] $\mathcal{HH}$-spaces admitting twisting null geodesic congruence*, Classical and Quantum Gravity **25**, 055010

[P2] Chudecki A. i Przanowski M., 2008, *From hyperheavenly spaces to Walker and Osserman spaces: I*, Classical and Quantum Gravity **25**, 145010

[16] The metrics which belong to this class and they admit the null Killing vector was considered earlier in [H4].

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[P3] Chudecki A. i Przanowski M., 2008, *From hyperheavenly spaces to Walker and Osserman spaces: II*, Classical and Quantum Gravity **25**, 235019

[P4] Przanowski M., Formański S. i Chudecki A., 2012, *Notes on para-Hermite-Einstein space-times*, International Journal of Geometric Methods in Modern Physics, Vol. 9, No. 1, 1250008

[P5] Chudecki A. i Przanowski M., 2013, *Killing Symmetries in $\mathcal{H}$ spaces with Λ*, Journal of Mathematical Physics **54**, 102503

[P6] Chudecki A. i Dobrski M., 2014, *Proper conformal symmetries in self-dual Einstein spaces*, Journal of Mathematical Physics **55**, 082502

[P7] Chudecki A., 2016, *All complex and real ASD Einstein spaces with Λ admitting nonnull Killing vector*, International Journal of Geometric Methods in Modern Physics, Vol. 13, No. 2, 1650011

[P8] Chudecki A., 2017, *Congruences of null strings and their relations with Weyl tensor and traceless Ricci tensor*, Acta Physica Polonica B Proceedings Supplement, Vol. 10, No. 2

[P9] Chudecki A. i Przanowski M., 2018, *On twisting type [N] $\otimes$ [N] Ricci flat complex spacetimes with two homothetic symmetries*, Journal of Mathematical Physics **59**, 042504

5.2 Papers [P1], [P9]

The papers [P1] and [P9] are devoted to the spaces for which both SD and ASD parts of the Weyl tensor are of the type [N]. These spaces are equipped with the congruences of SD and ASD null strings such that their intersection constitutes the congruence of null geodesic with twist. In notation proposed in [H8] the symbol for such spaces is $\{[\mathrm{N}]^e \otimes [\mathrm{N}]^e, [++]\}$. These are the generic complex spaces for the Lorentzian vacuum type [N] twisting spaces.

In [P1] we used the form of the key function for the expanding hyperheavenly space which was found in [49]. We inserted this key function into expanding hyperheavenly equation and we obtained the general equation (3.2) for the spaces of the types [deg] $\otimes$ [deg]. (Similar approach was proposed in [13, 19]). Then we specified Eq. (3.2) and we obtained the equation for the type [N] $\otimes$ [N] (Eq. (3.4)). We focused on the analysis of the special case of this equation. Finally we found an interesting example (the metric (4.15)). The metric (4.15) is equipped with the twisting congruence of null geodesics but it does not admit a Lorentzian slice. It belongs, however, to the class of the Walker spaces. It suggested that the hyperheavenly spaces formalism can be used in investigations of the Walker spaces ([P2] was devoted to this problem).

To find the twisting solutions of the type [N] $\otimes$ [N] spaces we used the hyperheavenly spaces formalism once again in [P9]. The first step was to find the key function for the spaces of the type [N] $\otimes$ [N] in a different form than the form proposed by J.F. Plebański and G.F. Torres del Castillo in [49]. We found slightly different coordinates than the coordinates used in [13, 19]. The key function in these coordinates has the form (3.12). The expanding hyperheavenly equation splits into the system of three equations (3.21) for two functions of three variables[17]. This system of equations is very complicated. We arrived at the conclusion that the analysis of the space of the types [N] $\otimes$ [N] with twist and without any symmetries is a very difficult task. We decided then to equip the space with two homothetic symmetries. We proved, that the Killing vector can be always brought to the form ∂_w and the homothetic vector can be brought to the form $w\partial_w + t\partial_t + (1-2\chi_0)(\phi\partial_\phi + \eta\partial_\eta)$.

The set of Eqs. (3.21) splits naturally in two branches. The generic one leads to the metric (6.6) and the field equations reduce to the extremely complicated, nonlinear ODE of

[17]This overdetermined system of equations is now intensively analyzed in the case with no symmetries and in the case with one homothetic symmetry.

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the fifth order (6.17). Unfortunately, we were not able to reduce this equation or find any special solution or even reconstruct the Hauser solution. However, we proved that vacuum type $[\mathrm{N}] \otimes [\mathrm{N}]$ twisting spaces equipped with two symmetries always have a solution. Such a solution exists for arbitrary initial conditions, for arbitrary value of the homothetic parameter χ_0 and for Lorentzian and neutral signatures of the metric. Moreover, our approach is a progress over the approach proposed in [14]. In [14] the field equations were not reduced to a single equation.

The generic case was very complicated, but the special case we were able to solve completely. The metric (5.17) constitutes the solution which depends on one function of one variable $U(h)$ which satisfies Eq. (5.18) with the solution given by the power series (5.16). Eq. (5.18) is very similar to the equation which appears in the Hauser solution. However, in Eq. (5.18) the homothetic parameter is arbitrary while in the Hauser solution it takes a special value. Detailed analysis proved that the space with the metric (5.17) is equipped with the nonexpanding congruence of ASD null strings. Consequently, it does not admit a Lorentzian slice. In notation proposed in [H8] it is the space of the type $\{[\mathrm{N}]^e \otimes [\mathrm{N}]^n, [++]\}$[18].

5.3 Paper [P2]

The hyperheavenly spaces formalism in investigations of geometry of the Walker spaces was used in the paper [P2]. The Walker space is defined as a triple $(\mathcal{M}, g, \mathcal{D})$ where $\mathcal{M}$ is n-dimensional smooth manifold, g is the pseudo-Riemannian metric and $\mathcal{D}$ is r-dimensional totally null and parallely propagated distribution [8, 27, 61]. In [P2] we focused on the case $n = 4$ and $r = 2$.

First we pointed out the relation between the hyperheavenly spaces and the Walker spaces. We defined *a weak hyperheavenly space* (see Definition 4.4, page 13) and we found the metric of such space (3.4). Then we proved that every weak and real hyperheavenly space is conformally equivalent to the Walker space. Using the spinorial formalism we found the metrics of the SD Walker spaces and SD Einstein-Walker spaces. Especially interesting is the case of SD Einstein-Walker space with $\Lambda = 0$. In this case we obtained the set of equations (4.31) (this set has been found earlier in [8]) which solution was unknown.

Further we considered the spaces equipped with two parallely propagated distributions. One of these distributions was SD and the second one was ASD. We defined *two-sided Walker space* and we found its general metric (Theorem 5.1). We used the results of the Theorem 5.1 to solve the set (4.31). This way we obtained the metric of the SD Einstein-Walker space with $\Lambda = 0$ explicitly.

5.4 Paper [P3]

The paper [P3] is the most transparent example of the use of the hyperheavenly spaces formalism in geometrical problems of the real manifolds. It is devoted to the pointwise and globally Osserman and Jordan-Osserman spaces. First we proved the relation between such spaces and the hyperheavenly spaces. The basic theorem [1] says that a space is a pointwise Osserman space if and only if it is self-dual (or anti-self-dual) Einstein space, i.e., it is space of the type $[\mathrm{any}] \otimes [-]$ or $[-] \otimes [\mathrm{any}]$.

We focused on algebraically degenerated pointwise Osserman spaces equipped with the expanding congruences of null strings, i.e., the spaces of the types $[\mathrm{deg}]^e \otimes [-]^e$. For such spaces the cosmological constant is necessarily nonzero[19]. The metrics were found explicitly (Theorem 3.1). It was the first time when the metrics of the Osserman spaces which were not the Walker spaces were found explicitly. The metrics for the globally Osserman, pointwise and globally Jordan - Osserman spaces were also presented.

[18]The example given in [P1] is the special case of the metric (5.17) in [P9].
[19]The case of the hyperheavenly spaces of the types $[\mathrm{deg}]^e \otimes [-]$ with $\Lambda = 0$ has been solved in [18].

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5.5 Paper [P4]

The paper [P4] is devoted to the complex para-Hermite Einstein spaces. The ways how to reduce vacuum Einstein equations with cosmological constant for the para-Hermite spaces are analyzed in [P4] in details. Para-Hermite spaces are equipped with two different congruences of SD (or ASD) null strings. Therefore the only possible types of such spaces are the types $[\text{any}] \otimes [\text{D}, -]$ (the orientation was chosen in such a manner that both congruences are ASD). In all the cases vacuum Einstein equations were reduced to a single equation. In the most general case (it is the space of the type $[\text{any}] \otimes [\text{D}]^{ee}$ with $\Lambda \neq 0$) field equations were reduced to the equation (5.13). The analysis of Eq. (5.13) became the foundation of the paper [H6]. One of the most interesting results is the fact that if para-Hermite Einstein space is equipped with one expanding and one nonexpanding congruence of ASD null strings then ASD part of the Weyl tensor vanishes. Consequently, the space reduces to the heavenly space of the type $[\text{any}] \otimes [-]$.

5.6 Papers [P5], [P7]

The papers [P5] and [P7] are devoted to the complex ASD spaces with the cosmological constant equipped with a symmetry defined by the Killing vector. This problem in the real spaces with the metric of the signature $(++++)$ was considered in [53, 59] (in such spaces only nonnull Killing vectors exist). Two different forms of the Killing vector admitted by the ASD spaces with the cosmological constant were found in [53]. In [59] it was proved that these two Killing vectors are, in fact, the same vector. In the real neutral spaces the problem of the nonnull Killing vectors was considered in [26].

The first step in [P5] consists of the proof that ASD Einstein spaces with the cosmological constant admit only Killing vectors. Then we reduced Killing equations to the single equation (3.16). We called this equation *the master equation*. However, the master equation for the ASD Einstein spaces is problematic because it contains the first integral of the heavenly equation with the cosmological constant. This quantity is denoted in [P5] by Υ. Υ is very complicated function. Therefore, there was no chance to solve the master equation. Fortunately, it appeared that the function Υ can be gauged away by the appropriate choice of the congruence of SD null strings [20].

The appropriate choice of the congruence of SD null strings appeared to be crucial in the problem considered. The metric of the heavenly space with the cosmological constant which admits the null Killing vector was explicitly found - it is the metric (4.22). This metric considered as a real one with the neutral signature metric is the general metric of the 4-dimensional globally Osserman space with nonzero curvature scalar admitting null Killing vector[21]. The case with the nonzero Killing vector appeared to be much more complicated. We proved, however, that in this case the field equations reduce to *the Boyer-Finley-Plebański (BFP)* equation (also called *Toda field equation*). The same reduction holds true in the real heavenly spaces with the metric signatures $(++++)$ and $(++--)$.

I returned to the problem of the nonnull Killing vectors in the heavenly spaces with the cosmological constant in the paper [P7]. The main aim of [P7] was to analyze in details the transformation which leads from the heavenly spaces in PRF coordinates (the metric (2.30)) to LeBrun coordinates [29] (the metric (2.34)). LeBrun coordinates were used in earlier works [26, 59]. Moreover, I found all real slices of the metric (2.34) and I proved a theorem (Theorem 2.1) a little more general then the theorem presented in [P5].

[20] Note, that the similar problem appeared in the heavenly spaces with $\Lambda = 0$ [17]. To solve this problem J.D. Finley i J.F. Plebański used *the complementary congruence of null strings*. However, in the heavenly spaces with $\Lambda \neq 0$ such trick does not work.

[21] The same metric has been found independently by M. Dunajski and P. Tod in [11].

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5.7 Paper [P6]

Proper conformal symmetries in Einstein spaces are rare in the sense that in the non-conformally flat Lorentzian spaces they are admitted only by the type [N] pp-wave metric. In [H1] the complex space of the type $\{[\mathrm{N}]^n \otimes [\mathrm{N}]^n, [--]\}$ equipped with the proper conformal vector was considered. Its Lorentzian slice is pp-wave metric. The only heavenly spaces which admit proper conformal symmetry are spaces of the type $[\mathrm{N}]^n \otimes [-]$. We devoted to such spaces the paper [P6].

We proved that there are two essentially different classes of the proper conformal symmetries in the spaces of the type $[\mathrm{N}]^n \otimes [-]$. We analyzed the geometric and algebraic differences between them. The geometric difference is related to the properties of the congruences of ASD null strings. The spaces of the type $[\mathrm{N}]^n \otimes [-]$ are equipped with infinitely many congruences of ASD null strings but two of them are distinguished (these are the congruences defined by the equations (2.18a)-(2.18b)). If both of these congruences are expanding we deal with more complicated class of the proper conformal symmetries (*Class II* in [P6]). If one of these congruences is nonexpanding we deal with the *Class I*. The algebraic difference is obvious. The Einstein field equations for the Class I were reduced to a single PDE of the first order which were solved completely. In this case we arrived at the metric (2.23) with the solution given by (3.2). The Einstein field equations for the Class II were reduced to a single PDE of the second order (Eq. (2.26) with constant $a_0 \neq 0$). The general solution of Eq. (2.26) is unknown. However, we presented an algorithm how to construct the solution using the only nonzero curvature scalar $C^{(1)}$. As a simple example of the metric which admits such a symmetry we proposed the metric (4.10).

5.8 Paper [P8]

In [P8] the most important results published later in [H7] were presented. However, one of the results is especially interesting and original (it has been published, for now, only in [P8]). This is the general metric (3.7) of the space which admits two nonexpanding congruences of SD null strings and one nonexpanding congruence of ASD null strings. This is the metric of the space of the type $[\mathrm{D}]^{nn} \otimes [\mathrm{II}]^n$. It belongs to the two-sided Walker class as well as to the para-Kähler class. The case of the type $[\mathrm{D}]^{nn} \otimes [\mathrm{II}]^n$ Einstein space was also completely solved. The metric of such space is given by (3.8). There appear four arbitrary functions of two variables in (3.8)[22].

5.9 Summary and future research

In the Theoretical Physics Group (team leader: prof. dr hab. Maciej Przanowski) I have been working since 2002. I published 17 articles. Three papers ([P1] - [P3]) were published before I received the Ph. D. degree (year 2009). In the papers published in 2010-2018 I dealt with the symmetries in the hyperheavenly spaces, geometry of the congruences of null strings, relation between symmetries and geometry of the congruences of null strings, geometry of the spaces equipped with two different congruences of null strings and the methods of reduction of the field equation in Einstein spaces.

My investigations allowed to find an answer for the unsolved problems formulated by the creators of the hyperheavenly spaces theory (J.F. Plebański and I. Robinson and their co-workers: J.D. Finley, C.P. Boyer, S. Hacyan, M. Przanowski and others). In the meantime I have found and defined a few other issues I am going to deal with in future. The most interesting problems are listed below.

[22] These two metrics are only the examples of more extensive work devoted to the two-sided Walker spaces and two-sided sesqui-Walker spaces (*sesqui-Walker* spaces have been defined in [28]). Such spaces are now under intensive investigations and they are one of my main scientific interests.

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- An analysis of Einstein equations for the hyperheavenly spaces of the type $\{[\mathrm{N}]^e \otimes [\mathrm{N}]^e, [++]\}$ with one homothetic symmetry and without any symmetries (this is the joint work with prof. dr hab. M. Przanowski).

- The explicit solutions of the para-Hermite Einstein spaces of the type $[\mathrm{D}]^{ee} \otimes [\mathrm{N}]^n$ (such spaces exist only if $\Lambda = 0$) and $[\mathrm{D}]^{ee} \otimes [\mathrm{N}]^e$, because I was not able to find the examples of such solutions in [H6]. On the other hand, I believe that the para-Kähler Einstein space of the type $[\mathrm{D}]^{nn} \otimes [\mathrm{N}]^e$ can be solved in all generality.

- The examples of two-sided Walker and two-sided sesqui-Walker Einstein spaces with the different properties of the congruences of null geodesics (the paper is ready in 70%).

- Subclassification of the congruences of null strings. Nowadays we distinguish only expanding and nonexpanding congruences of null strings so such classification is rather not very detailed. Interesting question arises if the properties of the Sommers vector can be used as an additional criterion of the classification of the congruences of null strings.

- An explicit solution of the space of the type $[\mathrm{I}]^{eeee} \otimes [\mathrm{any}]$ (approach proposed in [H7] is not satisfactory) and the space of the type $[\mathrm{I}]^{eeee} \otimes [\mathrm{I}]^{eeee}$ in the neutral and Lorentzian signatures. In Lorentzian case such a space is equipped with four different shearfree congruences of null geodesics. From the Goldberg - Sachs theorem it follows that such space is algebraically general (type [I]) or it is conformally flat. It cannot be the Einstein space. What are the possible types of traceless Ricci tensor in such space?

The existence of the congruences of null strings and the hyperheavenly spaces formalism allowed to solve many geometrical issues. Many new problems have been defined. The most transparent results concern the real neutral spaces. However, it must be emphasize that the main goal of our investigations is searching for general techniques of obtaining real Lorentzian slices of the complex spaces. We hope that detailed studies on the geometry of the null strings and their intersections allow to find new vacuum and algebraically special solutions of Einstein field equations.

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